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Discerning differences among anomalously wandering directed polymers

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We contrast the effects of uncorrelated power-law noise and linearly correlated Gaussian noise upon the anomalous wandering of directed polymers in random media. In the first instance, we explore the role of the noise on the morphology of the ultrametric tree structure of the ensemble of locally optimal paths, and then, motivated by the work of Zhang [Phys. Rev. Lett. **59**, 2125 (1987)] and more recently by Perlsman and Schwartz [Europhys. Lett. **17**, 11 (1992)], provide strong evidence for the universality of the model's river basin patterns. Lastly, we discuss those precise features of the positional and energy fluctuations that could permit a resourceful experimentalist to discern the noise distribution underlying anomalous roughening recently observed in fire fronts, bacterial growth, and fluid flow through porous media.

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Because of its immediate and varied connections to a broad collection of outstanding, difficult questions in the statistical mechanics of ill-condensed matter [1], the problem of directed polymers in a random medium (DPRM) has achieved a well-deserved notoriety, capturing the attention of many practitioners in the field. In its initial guise, the $(1+1)$-dimensional DPRM appeared in the context of impurity-induced domain-wall roughening in disordered two-dimensional (2D) magnets [2]. More recently, the many-dimensional DPRM has surfaced in discussions of Abrikosov vortices in high-T_c superconductors [3], dislocation lines in disordered solids [4], as well as the possible conformations of a polyelectrolyte in a gel matrix. Kardar and Zhang [5], who introduced the DPRM and noted its important connection to the noisy Burgers equation, made the crucial observation that the physics of the low-dimensional DPRM was controlled by a zero-temperature fixed point, implying that the finite-temperature properties of the polymer would be identical to those at $T=0$. These same authors, with the assistance of the Europeans [6], soon came to appreciate the DPRM as a more manageable but equally rich version of the very stubborn spin-glass problem, complete with issues of replica-symmetry breaking (RSB), and structural matters associated with an ultrametric free-energy landscape. Indeed, at $T=0$, the DPRM becomes a matter of global optimization, an amusing variant of the traveling-salesman problem in which, fittingly enough, maximization of commission, rather than minimization of path length, becomes the goal at hand. Finally, as stressed by Kardar and Zhang, the ultrametric properties of the DPRM ensemble of locally optimal paths bear a striking resemblance to patterns produced by Nature herself, including river basin deltas, capillary networks in the circulatory system, and neuronal arrays in the brain.

It is the purpose of this paper to concentrate on these geometric properties; not so much with an eye to testing their legitimacy as models of Nature's pattern formation, but rather on the role of the noise distribution on their structure. In particular, we concern ourselves with power-law noise distributions which, within the realm of KPZ [7], have recently been proposed to account for *anomalous scaling* [8] observed in fire fronts [9], bacterial colonies [10], and fluid fronts in porous media [11]. In the present context, the freedom associated with the power-law tail of the noise distribution permits us an interesting laboratory in which to study the changing morphology of the random energy landscape, to make connections to the apparently unrelated case of correlated, Gaussian noise [12], to comment on recent findings [13] regarding DPRM ultrametricity, and lastly to discuss, precisely, what are the discernable differences between anomalous wanderers.

Let us first recall the $T=0$ formulation of the $1+1$ DPRM. It is conveniently visualized [14] by considering the first quadrant of the square lattice, upon which quenched random energies (drawn from a variety of probability distributions—uniform, Gaussian, power-law) are placed on all the bonds. Neighboring bonds are assumed, momentarily, to be uncorrelated. It is in this manner that we create a single realization of the random energy landscape. Our interest is in the collection of paths emanating from the origin that proceed forward by single steps in the vertical (y) or horizontal (x) directions. As these paths are directed (backtracking not permitted), we naturally identify the diagonal ($y=x$) as the longitudinal or time direction. The energy of a given path of n steps is given by the algebraic sum of the bonds visited along the way. There are 2^n paths reaching the end points of the nth time slice and, of these, $n+1$ constitute the ensemble of locally optimal paths of maximal energy with end point specified. Geometrically, this set (see Fig. 1, for example) gives rise to the DPRM *river basin deltas* mentioned earlier. Note, however, that the details of the pattern are very much sample dependent, though interesting statistical information can be had and, furthermore, even for a single random energy landscape, configurations undergo drastic revision from one time slice to the next. Within a locally optimal ensemble, the path of absolute greatest energy is referred to as *globally optimal*. The statistical properties of this sole path, obtained by averaging over many realizations of the random energy landscape, are well known for the 1+1 DPRM. Its transverse positional fluctuations about the symmetry axis are governed by the wandering exponent ζ, which, for uncorrelated Gaussian noise [2], has the superdiffusive value $\frac{2}{3}$.

Our interest at this point is in power-law random bond energies distributed as $P(\varepsilon)=\varepsilon^{-(\mu+1)}$ for $\varepsilon>1$, zero otherwise, as first proposed by Zhang [8,15]. We generated the power-law tails by drawing a random number uniformly from the unit interval [0,1] and raising it to the power $-1/\mu$, being careful to be aware of the graininess of the *Numerical Recipes* [16] supplied subroutine RAN3. This is especially important for the power-law DPRM because it is the rare event, buried deep in the tail, that fixes the scaling properties, both energy and geometric. Our main purpose, initially, was to understand the effect of varying μ on the nature of the random-energy landscape and, in particular, its role in determining the geometric structure of the DPRM delta. As is well known, the case $\mu=2$ is marginal (the second moment of the bare probability distribution is logarithmically divergent), with $\mu<2$ corresponding to Levy flights, while $\mu\approx 2-3$ has become the focus of attention of the kinetic roughening community as the most physically relevant range, providing a possible explanation for the anomalous scaling [9–11] observed in propagating fire fronts, bacterial colonies, and fluid fronts in porous media. It should be stressed that, in sharp contrast to the DPRM with Gaussian noise (both correlated and not), there has been precious little analytical progress in the context of power-law distributions, aside from a novel Flory argument proposed independently by Krug [17] and Zhang [18] that is based on the occurrence of rare, but dominant fluctuations inherent in power-law tails. These authors predict $\zeta(\mu)=(\mu+1)/(2\mu-1)$ for the power-law DPRM wandering exponent—a formula that is entirely consistent with numerical simulations for small values of $\mu>2$, but incorrectly predicts a $\mu_c=5$. Indeed, recent work of Bourbonnais, Herrmann, and Vicsek [19] has convincingly established that it is for $\mu_c=7$ that the power-law tail drops sufficiently fast for the wandering exponent to fall to its uncorrelated, Gaussian noise value.

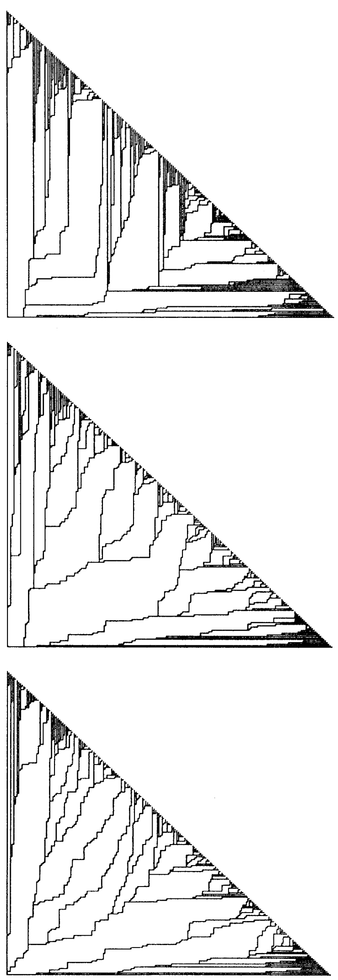

FIG. 1. River basin deltas associated with the power-law DPRM: (a) $\mu=1$, (b) $\mu=3$, and (c) $\mu=7$. The patterns were all generated from the same seed, but evolve differently when the tail of the distribution is changed.

In Fig. 1, we plot the DPRM deltas for $\mu=1$, 3, and 7. Please note that each of the deltas was generated using the same seed to initialize the random number generator, so that the figure truly illustrates the evolution of the river basin morphology as the power-law tail is changed. We observe the following features. For $\mu=1$, which corresponds to an extreme Levy flight case, there is a predominance of straight-line segments and persistence of right angle vertices on *macroscopic scales* that appear lost, or at least greatly weakened, as μ is increased beyond 2 to 3. This substantive qualitative change is also manifest as $\mu\rightarrow 3$ in an increased jaggedness of the trajectories and an enhanced meandering in the middle of the delta. By contrast, a comparison of the deltas for $\mu=3$ and 7 reveals a morphological shift that is much more modest, being one of degree rather than kind. In other words, the important geometric characteristics gleaned from the $\mu=3$ delta are maintained, and merely amplified, as the range of the power-law tail is decreased. Finally, there is a growing ratio of black to white (indication of an increasing fractal dimension), while the persistance of linearity is for the most part relegated to the border paths and their immediate neighbors.

In Fig. 2, we have paired together the DPRM delta corresponding to power-law noise with $\mu=2$, for which numerical studies strongly suggest that $\zeta=1$, and the river basin pattern generated by linearly correlated, Gaussian noise. For the latter, the globally optimal path through the random energy landscape is identical to the 1D interface in a 2D random-field (RF) Ising model, a problem [20] with a long, colorful, and contentious history that is, nevertheless, characterized by the very same wandering exponent as $\mu=2$ power-law noise. That $\zeta_{\rm RF}=1$ was established numerically by Fernandez *et al.*

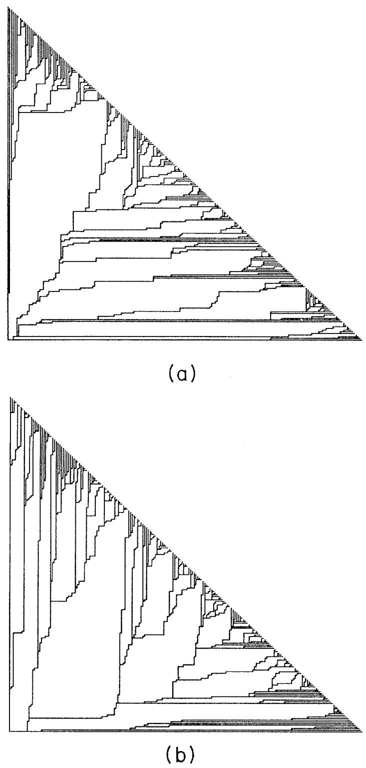

FIG. 2. River basin deltas resulting from two very different bare noise distributions that, nevertheless, produce identical DPRM wandering exponents. (a) linearly correlated, Gaussian noise; (b) uncorrelated, $\mu=2$ power-law noise. Distinct random number generator seeds; apparently quite different patterns.

[21], then analytically by Zhang [22] via the stochastic Burgers equation, and finally rigorously by Imbrie [23], who settled once and for all the lower critical dimension of the model. Unfortunately, the fact that two grossly different noise distributions can give rise to the same anomalous wandering exponent has muddied the waters somewhat within the kinetic roughening community, though the presently prevailing sentiment is to attribute the anomalous scaling seen in the experiments [9–11] to uncorrelated power-law noise distributions. From a materials science point of view, it is plausible, though not entirely obvious, why this should be so, but, in an effort to categorize what, if any, are the discernible differences between the RF and $\mu=2$ distributions upon the scaling properties of anomalous wanderers, we felt obliged to address this matter, both at the geometric level of the DPRM deltas, as well as the fully renormalized probability distributions.

With regard to delta ultrametricity, our findings strongly support the notion of universality first proposed by Zhang [14], who phrased the discussion of basin geometry in terms of ancestry and progeny of the network, as well as the more recent efforts of Perlsman and Schwartz [13], who addressed this issue quantitatively by examining the cross section and branching probability of the globally optimal path. For the special case of uncorrelated, Gaussian noise, these authors provide quite convincing numerical evidence that all geometric properties of the basin pattern are dictated by the wandering exponent ζ. Our findings are in complete agreement with this idea—Fig. 3 reveals that the large sample-dependent differences evident in the deltas of the previous figure are statistically irrelevant, so that upon averaging over many realizations of the random-energy landscape, an important geometrical property of the delta, such as the cross section of the globally optimal path, scales the same for both the $\mu=2$ and RF distributions. The curves have slopes identical to our numerically determined wandering exponents, both near unity. Note that the cross section, defined as the total number of branches emanating from the globally optimal path, scales properly only over a limited range, as first observed by Perlsman and Schwartz [13]. The abrupt saturation that occurs at large heights is easily understood in a quantitative fashion as a manifestation of a (non)-interference effect seen by Mezard [6] in a related context. Additional evidence in favor of universality is given in Fig. 4, which shows that the branching probability of the globally optimal path is apparently the same regardless of the distribution. Thus, our own conclusion is clear—at the level of the DPRM delta, there are no discernible differences, since the important statistical properties of the basin geometry are dictated entirely by the value of wandering exponent, not the underlying, bare probability distribution.

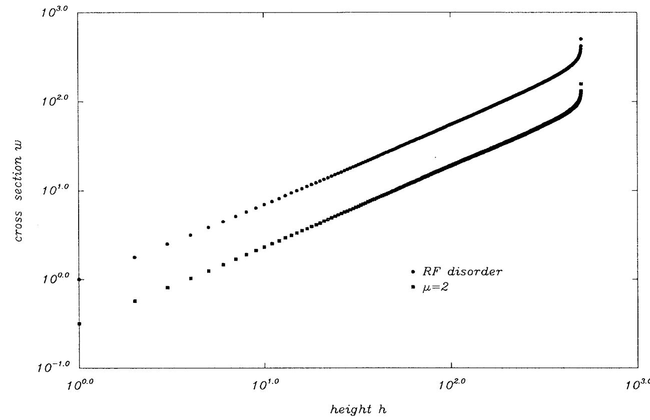

FIG. 3. *Cross section* of the globally optimal path as a function of height h from the base of the river basin delta. At a particular height, w is defined as the total number of rivulets that emanate from the globally optimal path below that point. 5000 realizations of the random-energy landscape. The data strongly suggest that the RF and $\mu=2$ cases scale in the same manner, with slopes identical to our independently extracted wandering exponents, as predicted by Perlsman and Schwarz [13]. Lower curve offset -0.5 for visibility.

Dramatic differences between RF and $\mu=2$ anomalous wandering are easily perceived, however, when we ask appropriate questions at the level of probability distributions [24] and amplitudes [25] that *completely* characterize the fluctuations of the globally optimal path. For example, the relative amplitudes of the first four even moments of positional probability distribution are in the ratio 1.00:1.30:1.52:1.70 for the $\mu=2$ case, but 1.00:1.24:1.40:1.53 for the RF DPRM. This rather impressive distinction appears to arise from the distinction

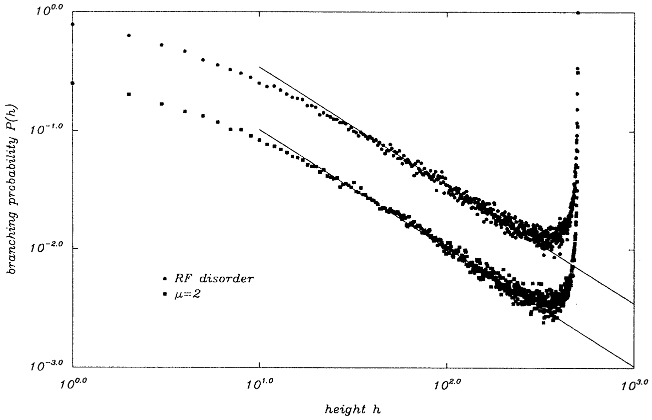

FIG. 4. *Branching probability* of the globally optimal path as a function of height for RF and $\mu=2$ DPRM. $P(h)$ is the normalized probability, obtained by sampling 5000 realizations of the random-energy landscape, that the globally optimal path branches at a height h above the base of the river basin delta. The straight lines have slope -1, providing strong evidence that, aside from crossover and saturation effects, this probability falls inversely with height, independent of the underlying noise distribution. Perlsman and Schwartz [13] had previously noted this behavior for the uncorrelated, Gaussian DPRM. Our work establishes the universality of their discovery. Because of nearly perfect coincidence, the lower curve has been offset.

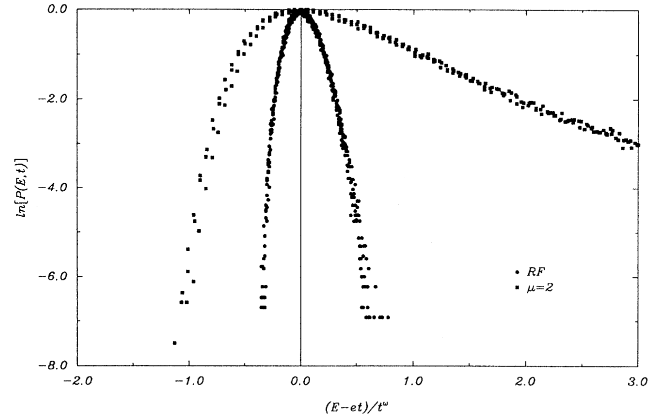

FIG. 5. Very different disorder-averaged energy probability distributions for the RF and $\mu=2$ DPRM. Here, $P(E,t)$ represents the probability that a globally optimal path of length t steps has total energy E. 50 000 realizations; path length 500 steps. Rescaled data collapse for time slices $t=300,400,500$. The energy fluctuation exponent ω is near unity for both cases, while e is the average energy per step.

between correlated and uncorrelated noise, since the ratios for various power-law DPRM all tended to be quite close, somewhat surprisingly, to the ordinary *random-bond* DPRM [24]. Differences are even more explicit for the renormalized energy probability distributions, see Fig. 5, where we note a narrow, statistically smooth, but highly skewed distribution for RF roughening (skewness $|s|=0.43$, compared to $|s|=0.29$ for the standard random-bond DPRM [25]). By contrast, though still asymmetric, the renormalized energy distribution resulting from a $\mu=2$ power-law tail is very noisy, with an ill-defined third cumulant, and much, much broader than that of the RF DPRM. As the fluctuations of the DPRM are closely tied to those of a stochastically evolving KPZ surface, we suggest the experimental relevance of these features in identifying the underlying noise distribution responsible for recently observed anomalous kinetic roughening. It is our expectation that the matter will soon be settled at this level of inquiry.

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